If the efficiency of firms varies neutrally,[1] as indicated by the error term in (1), and the prices paid for factors vary from firm to firm, then the levels of input are not determined independently but are determined jointly by the firm's efficiency, level of output, and the factor prices it must pay. In short, a fitted relationship between inputs and output is a *confluent* relation that does not describe the production function at all but only the net effects of differences among firms. (For a more general discussion, see [13, 15].)

In such cases, however, it may be possible to fit the *reduced form* of a system of structural relations such as (1) and (3) and to derive estimates of the structural parameters from estimates of the reduced-form parameters. Not only does it turn out to be possible in this case, but an important reduced form turns out to be the cost function:

$$c = ky^{1/r} p_1^{a_1/r} p_2^{a_2/r} p_3^{a_3/r} v \,, \tag{4}$$

where

$$k = r(a_0 a_1^{a_1} a_2^{a_2} a_3^{a_3})^{-1/r} \,,$$

$$v = u^{-1/r} \,,$$

and

$$r = a_1 + a_2 + a_3 \,.$$

The parameter r measures the degree of returns to scale. The fundamental duality between cost and production functions, demonstrated by Shephard [17], assures us that the relation between the cost function, obtained empirically, and the underlying production function is unique.[2] Under the cost minimization assumption, they are simply two different, but equivalent ways of looking at the same thing.

Note that the cost function must include factor prices if the correspondence is to be unique. The problem of changing (over time) or differing (in a cross section) factor prices is an old one in statistical cost analysis; see [10, pp. 170-76]. Most generally, it seems to have been handled by deflating cost figures by an index of factor prices, a procedure that Johnston [10] shows typically leads to bias in the estimation of the cost

[1] A model incorporating non-neutral variations in efficiency of the form

$$y = (a_0 u_0) x_1^{a_1 u_1} x_2^{a_2 u_2} x_1^{a_3 u_3}$$

was discussed in my paper "On Measurement of Relative Economic Efficiency," abstract, *Econometrica*, **28** (July 1960), 695. It is interesting to note that despite the complex way in which the random elements u_0, u_1, and u_2 enter, there are circumstances under which it is possible to estimate the parameters in such a production function.

[2] I owe this point to Hirofumi Uzawa. It is true, of course, only if all firms have the same production function, except perhaps for differences in the constant term, so that aggregation difficulties may be neglected.

curve unless correct weights, which depend on (unknown) parameters of the production function, are used. It seems strange that no one has taken the obvious step of *including factor prices directly in the cost function.* If price data are available for the construction of an index and prices do not move proportionately, in which case no bias would result from deflation, why not use the extra information afforded?

What form of production function is appropriate for electric power? The generalized Cobb-Douglas function presented above is attractive for two reasons: First, it leads to a cost function that is linear in the logarithms of the variables

$$C = K + \frac{1}{r} Y + \frac{a_1}{r} P_1 + \frac{a_2}{r} P_2 + \frac{a_3}{r} P_3 + V, \tag{5}$$

where capital letters denote logarithms of the corresponding lower-case letters. The linearity of (5) makes it especially easy to estimate. Second, a single estimate of returns to scale is possible (it is the reciprocal of the coefficient of the logarithm of output), and returns to scale do not depend on output or factor prices. (The last-mentioned advantage turns out to be a defect as we shall see when we come to examine a few statistical results.) But does such a function accurately characterize the conditions of production in the electric power industry?

A casual examination of trade publications suggests that once a plant is built, fixed proportions are more nearly the rule. Support for this view is given by Komiya [11], who found that data on inputs and output for individual plants were better approximated by a fixed-proportions model that allowed differences in the proportions due to scale. A simplified version of Komiya's model is[3]

$$\begin{aligned} x_1 &= a_1 y^{b_1}, \\ x_2 &= a_2 y^{b_2}, \\ x_3 &= a_3 y^{b_3}. \end{aligned} \tag{6}$$

At the firm level, however, there are many possibilities for substitution that may go unnoticed at the plant level; for example, labor and fuel may be substituted for capital by using older, less efficient plants more intensively or by using a large number of small plants rather than a few large ones.

[3] Since y is exogenous, it would be appropriate to estimate the coefficients in (6) by least squares. An objection to this, however, is the fact that, if individual plants are considered, the output allocated to *each* is not exogenous; see Westfield [19, pp. 15-81). Furthermore, Komiya does not use output but name-plate rated capacity and input levels adjusted to full capacity operation. It is even more doubtful whether the former can be considered as exogenous in a cross section. My objection here is closely related to the one raised by Hughes (see p. 169); however, while the endogenicity of output at the plant level is clear, its endogenicity at the firm level for a member of a power pool is conjectural.

Given persistent differences in the factor prices paid by different firms, cross-section data should reflect such possibilities of substitution. Certainly, as a provisional hypothesis, a generalized Cobb-Douglas function may be appropriate.

It would, of course, be preferable to *test* whether significant substitution among factors occurs at the firm level. The use of the generalized Cobb-Douglas unfortunately does not permit us to do so except in a very general way, since its form implies that the elasticity of substitution between any pair of factors is one. A more general form, which has both the Cobb-Douglas and fixed coefficients as limiting cases, has recently been suggested by Arrow, Minhas, Chenery, and Solow [1]. Constant returns to scale are assumed, but the form can be easily generalized; in a more general form it is

$$y = [a_1 x_1^b + a_2 x_2^b + a_3 x_3^b]^{1/f} . \tag{7}$$

In this case returns to scale are given by the ratio b/f and the elasticity of substitution between any pair of factors can be shown to be $1/(1 - b)$. In the special case in which $b = f$ it can be shown that the limiting form of (7) as the elasticity of substitution goes to zero is

$$y = \min \left\{ \frac{x_1}{(a_1 + a_2 + a_3)^{1/b} - 1}, \frac{x_2}{(a_1 + a_2 + a_3)^{1/b} - 1}, \frac{x_3}{(a_1 + a_2 + a_3)^{1/b} - 1} \right\}, \tag{8}$$

or fixed coefficients, and the limiting form as the elasticity of substitution goes to one is

$$y = (a_1 + a_2 + a_3)^{1/b} x_1^{a_1/(a_1+a_2+a_3)} x_2^{a_2/(a_1+a_2+a_3)} x_3^{a_3/(a_1+a_2+a_3)} , \tag{9}$$

or Cobb-Douglas. Although I have not formally demonstrated the fact, it is possible that the limiting form of the more general case (7) is something like the Komiya model as the elasticity of substitution tends to zero, and like the generalized Cobb-Douglas as it tends to one.

Unfortunately, in its generalized form (7) is quite difficult to estimate from the data available. Furthermore, although clearly superior to the generalized Cobb-Douglas form, (7) still implies that the elasticity of substitution between any pair of factors (e.g., labor capital and fuel capital) is the same, which hardly seems reasonable. Other generalizations are possible, but none that I have found thus far offers much hope of being amenable to a reasonable estimation procedure.

If the generalized Cobb-Douglas form is adopted, however, relatively simple estimation procedures can be devised for evaluating the parameters of the production function. The reduced form of (1) and (3) that incorporates all but one of the restrictions on the parameters in the derived demand equations (which are the more usual reduced form) is nothing but the cost function.

The only restriction not incorporated in (4) or (5) is that the coefficients of the prices must add up to one. It is a simple matter to incorporate this restriction, however, by dividing costs and two of the prices by the remaining price (it doesn't matter either economically or statistically which price we choose). When fuel price is used as the divisor, the result is

$$(10)\qquad C - P_3 = K + \frac{1}{r} Y + \frac{a_1}{r}(P_1 - P_3) + \frac{a_2}{r}(P_2 - P_3) + V,$$

which will be called Model A.

Model A assumes that we have relevant data on the "price" of capital and that this price varies significantly from firm to firm. If neither is the case, we are in trouble. Most of the results presented here are based on Model A, but the data used for this price of capital are clearly inadequate. (See Appendix B.) If one supposes, however, that the price of capital is the same for all firms, which is not implausible, one can do without data on capital price and use the restriction on the coefficients of output and prices to estimate the elasticity of output with respect to capital input. The assumption that capital price is the same for all firms implies

$$(11)\qquad C = K' + \frac{1}{r} Y + \frac{a_1}{r} P_1 + \frac{a_3}{r} P_3 + V,$$

where $K' = K + (a_2/r)P_2$, since the exponents of the input levels in (1) are assumed to be the same for all firms. Equation (11) is called Model B.

2. Some Statistical Results and Their Interpretation

Estimation of Model A from a cross section of firms requires that we obtain data on production costs, total physical output, and the prices of labor, capital, and fuel for each firm; for Model B we do not need the price of capital, since it is assumed to be the same for all firms. Details of the construction of these data for a sample of 145 privately owned utilities in 1955 are given in Appendix B and are not discussed here at any length. Suffice it to say that these data are far from adequate for the purpose, and I now believe that a better job could have been done with other sources.

The results from the least-squares regression suggested by equation (10) are given in line I of Table 3; the interpretation of these results in terms of the parameters of the production function is given in line I of Table 4. The R^2 is 0.93, which is somewhat unusual for such a large number of observations; increasing returns to scale are indicated, and the elasticities of output with respect to labor and fuel have the right sign and are of plausible magnitude; however, the elasticity of output with respect to capital price has the wrong sign (fortunately, it is statistically insignificant).

TABLE 3

RESULTS FROM REGRESSIONS BASED ON MODEL A FOR 145 FIRMS IN 1955

Regression No.	Coefficient				R^2
	Y	$P_1 - P_3$	$P_2 - P_3$	x	
I	0.721 (±.175)	0.562 (±.198)	−0.003 (±.192)	—	0.931
II	0.696 (±.173)	0.512 (±.199)	0.033 (±.185)	−0.046 (±.022)	0.932
IIIA	0.398 (±.079)	0.641 (±.691)	−0.093 (±.669)	—	0.512
IIIB	0.668 (±.116)	0.105 (±.275)	0.364 (±.277)	—	0.635
IIIC	0.931 (±.198)	0.408 (±.199)	0.249 (±.189)	—	0.571
IIID	0.915 (±.108)	0.472 (±.174)	0.133 (±.157)	—	0.871
IIIE	1.045 (±.065)	0.604 (±.197)	−0.295 (±.175)	—	0.920
IVA	0.394 (±.055)			—	
IVB	0.651 (±.189)			—	
IVC	0.877 (±.376)	0.435 (±.207)	0.100 (±.196)	—	0.950
IVD	0.908 (±.354)			—	
IVE	1.062 (±.169)			—	

Figures in parentheses are the standard errors of the coefficients.

The dependent variable in all analyses was $C - P_3$.

The variables are defined as follows:

C = log costs Y = log output P_1 = log wage rate P_2 = log capital "price"

$$P_3 = \text{log fuel price} \qquad x = \left| \frac{\text{output 1955} - \text{output 1954}}{\text{output 1954}} \right| .$$